CHARACTERISTIC OUTFLOW BOUNDARY CONDITIONS FOR SIMULATIONS OF ONE-DIMENSIONAL HEMODYNAMICS

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Abstract—The Windkessel outflow models, describing the relation between blood pressure and flow, are commonly used for modeling the truncated downstream vasculature bed in the arterial system. By means of a characteristic approach, a unified formalism of characteristic boundary conditions was proposed in the present work. A novel defined parameter, termed the reflection coefficient, is convenient to quantify and analyze the incoming waves occurring at the boundary of the computational domain. Numerical examples using different forms of outflow models including one-, two-, and three-element Windkessel models were studied and discussed. The results indicate that when the 1-D terminal ends are coupled with the three-element Windkessel outflow model, the system is able to capture the physiological features.

Keywords—Windkessel models, reflection coefficient, hyperbolic system, distributed arterial network, nonreflecting boundary condition.

I. INTRODUCTION

Pulse waves propagating along vascular segments have been modeled using one-dimensional (1-D) flow conservation laws of mass and momentum augmented by a structural state equation representing vessel wall mechanics. Experimental [1], [2] and clinical validation works [3] have shown that the distributed 1-D models of the arterial tree are sufficient to capture the pulse wave propagation phenomenon in large vessels.

Large arteries gradually bifurcate into smaller arteries from generation to generation, resulting in hundreds of millions of arterioles and capillaries being embedded in the tissue. In practice, it is not possible to exhaust this tree-like vasculature with millions of connected 1-D flow branches. The 1-D flow modeling of an arterial tree structure has to be terminated at certain finite number of bifurcation generations.

Nowadays, Windkessel models [4], lumped-parameter models (0-D) [5], [6], and structured tree models [7-9] have been used to represent the overall effect contributed by the truncated downstream vasculature bed. No matter what type of Windkessel model (two-element, three-element, etc) is used, according to the theory of characteristics, only one single boundary condition can be specified in the 1-D hyperbolic system [10].

The combination of mathematical requirement and physiological principle inspires the present concept. A special characteristic boundary condition treatment is devised in the present work. By introducing definition of the reflection coefficient, the outflow conditions are generalized into a unified formalism. This new form of boundary condition representation is particularly useful in the study of vascular hemodynamics. Single resistance, two-element, and three-element Windkessel models can all be accounted for using an appropriately specified reflection coefficient.

II. METHODOLOGY

A. Governing Equations for One-Dimensional Blood Flow

The blood flow in a tubular duct has been modeled by taking the cross-sectional average of the flow variables represented by Navier-Stokes equations in three dimensions, with the assumption that the flow is incompressible, Newtonian, axisymmetric, and laminar. Besides, pressure is a function of cross-sectional area only and the velocity profile is flat but with a boundary layer of thickness next to the wall. The resulting one-dimensional (1-D) equations in the longitudinal direction can be expressed as [7], [8]:

\[ \frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = S \]  

(1)
\[ Q = \begin{bmatrix} A \\ q \end{bmatrix}, \quad F = \begin{bmatrix} q \\ q^2 + B \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ -2\pi vqR \frac{\partial B}{\partial r} + \frac{\partial r}{\partial t} \end{bmatrix} \] (2)

where \( A \) is the lumen cross-sectional area of the tube with radius \( r \), \( q \) is the flow rate, \( \rho \) is the blood density, \( v \) is the kinematic viscosity (assumed to be constant), \( \delta \) is the boundary layer thickness (assumed to be constant), and \( p \) is the static pressure. The independent variables \( x \) and \( t \) represent, respectively, the spatial and time coordinates of the postulated 1-D governing equations.

To close up the above flow equations having three state variables \((A, q, \text{ and } p)\), an inclusion of a third equation, namely the state equation of the tube structure:

\[ p(x,t) - p_0 = \frac{4}{3} \rho \left(1 - \frac{A_0(x)}{A(x,t)} \right) \] (3)

where \( p_0 \) is the ambient or tissue pressure, \( A_0 \) is the cross-sectional area with radius \( r_0 \) at zero transmural pressure (i.e., at \( p = p_0 \)), and \( E \) and \( h \) are the Young’s modulus and thickness of the vascular wall, respectively.

**B. Characteristic Analysis**

The characteristic forms of Eqs. (1) and (2) are obtainable according to the characteristics theory. Following the work of Thompson [10], the present characteristic formalism can be expressed as:

\[ \begin{align*}
\frac{\partial W}{\partial t} + \lambda_1 \frac{\partial W}{\partial x} + S_1 &= 0 \\
\frac{\partial W}{\partial t} + \lambda_2 \frac{\partial W}{\partial x} + S_2 &= 0
\end{align*} \] (4)

where

\[ \lambda_{1,2} = \frac{q}{A} \pm c \] (5)

and

\[ \begin{align*}
L &= \frac{1}{2c} \begin{bmatrix} \lambda_2 & 1 \\
-\lambda_1 & -1 \end{bmatrix} \] (7)

which are called the characteristic or Riemann differences. At the boundary cells, the spatial derivatives \( \mathcal{L} \) need to be approximated using upwind differencing in order to comply with the direction of wave propagation. In addition, one particular physical boundary condition must be specified to replace the incoming wave mode of these two \( \mathcal{L} \):

\[ \begin{align*}
\mathcal{L} &= \lambda_1 \frac{\partial W}{\partial x} = \frac{\lambda_1}{2c} \left( -\lambda_1 \frac{\partial A}{\partial x} + \frac{\partial q}{\partial x} \right) \\
\mathcal{L} &= \lambda_2 \frac{\partial W}{\partial x} = \frac{\lambda_2}{2c} \left( \lambda_2 \frac{\partial A}{\partial x} - \frac{\partial q}{\partial x} \right)
\end{align*} \] (8)

where the subscript in \( \mathcal{L} \) represents the \( i \)-th wave mode propagating with characteristic velocity \( \lambda_i \). By using the transformation between characteristic and physical (primitive) variables, the physical variables \( A, q, \text{ and } p \) can be expressed in terms of \( \mathcal{L} \) as:

\[ \begin{align*}
\frac{\partial A}{\partial t} + \mathcal{L} + \mathcal{L} &= 0 \\
\frac{\partial q}{\partial t} + \lambda_1 \mathcal{L} + \lambda_2 \mathcal{L} - S_1 &= 0 \\
\frac{\partial p}{\partial t} + \frac{\partial c^2}{\partial A} (\mathcal{L} + \mathcal{L}) &= 0
\end{align*} \] (9)

At the 1-D outflow boundary, waves with \( \lambda_1 > 0 \) are outgoing waves, where the one-sided difference obtained using interior stencil points is adopted for evaluating the corresponding \( \mathcal{L} \). Waves with \( \lambda_2 < 0 \) are incoming waves, where the proper values for computing \( \mathcal{L} \) depend on how the physical boundary conditions are specified. In most physiological conditions, the blood flow velocity is smaller in magnitude than the wave velocity. This suggests that only one boundary condition needs to be imposed at the outflow boundary.

**C. Windkessel Outflow Models**

Windkessel models, commonly called 0-D or lumped parameter models, are simplified models that attempt to approximate the overall vascular effect of the lumped block. The flow motion characteristics representing inertia, volume dilation, and impedance to flow are modeled respectively using basic elements including conductance, compliance (or capacitance), and resistance. These basic elements can be assembled.
in-series or in-parallel to result in different models. The one-, two-, and three-element Windkessel models that have been suggested in the Windkessel family [4, 11] are described below.

1) Single Resistance (R) Model: A single resistance model is connected to the end of the 1-D model, as depicted in Fig. 1(a). The pressure gradient that drives the flow passing through the resistance obeys the flow-resistance relationship:

\[ q = \frac{p - p_\infty}{R} \]  \hspace{1cm} (10)

Taking the derivative of Eq. (10) with respect to time yields:

\[ \frac{dq}{dt} = \frac{1}{R} \frac{dp}{dt} \]  \hspace{1cm} (11)

2) Two-Element (CR) Windkessel Model: A commonly used lumped parameter model of the arterial system is the two-element Windkessel model, first mathematically formulated by Otto Frank in 1899 [12]. To describe the elastic blood vessel as a compliant compartment that stores blood in systole and recoils in diastole, a compliance \( C \) is installed in front of the resistance \( R \) of the single resistance model, as shown in Fig. 1(b).

\[ \frac{dp}{dt} = \frac{q - p - p_\infty}{C + RC} \]  \hspace{1cm} (12)

3) Three-Element (RCR) Windkessel Model: Adding a second resistance to the CR model leads to the creation of the three-element Windkessel model [4], as shown in Fig. 1(c).

\[ \frac{dq}{dt} = \frac{1}{R_1} \left( \frac{dp}{dt} - \frac{dp_{WK}}{dt} \right) \]  \hspace{1cm} (13)

where \( \frac{dp_{WK}}{dt} \) is governed by \( C \) and \( R_2 \) elements and defined as:

\[ \frac{dp_{WK}}{dt} = \frac{q}{C} \left( 1 + \frac{R_1}{R_2} \right) \frac{p - p_\infty}{R_2C} \]  \hspace{1cm} (14)

The RCR model is coupled to the 1-D tube end (see Fig. 1(c)) by combining Eqs. (9) and (13) in terms of \( \mathcal{Z}_1 \), \( \mathcal{Z}_2 \), with the use of \( S_2 = 0 \), \( \lambda_1 = c \), and \( \lambda_2 = -c \) in Eq. (9):

\[ \mathcal{Z}_2 = R_1 \mathcal{Z}_2 - \frac{1}{c(R_1 + Z_0)} \frac{dp_{WK}}{dt} \]  \hspace{1cm} (15)
where \( dp_{wk} / dt \) is defined in Eq. (14) and

\[
R_f = \frac{R_1 - Z_0}{R_1 + Z_0}
\]  

(16)

is called reflection coefficient [13].

The aforementioned Windkessel models (R, CR, and RCR) can be obtained by specifying different parameters in Eq. (15). For example, if \( C = 0 \) and \( R_f = 0 \), and the time-dependent term \( dp_{wk} / dt \) vanishes, the boundary condition reduces to the partially reflecting R model (see Fig. 1(a)):  

\[
L_2 = R_f L_1
\]  

(17)

If \( R_f = 0 \), the boundary condition recovers the CR model (see Fig. 1(b)):  

\[
L_2 = -L - \frac{A}{\rho c^2} \left( \frac{q}{C} - \frac{p - p_o}{RC} \right) = -L - \frac{1}{cZ_0} \frac{dp}{dt}
\]  

(18)

Finally, if \( R_f = Z_0 \) (i.e., \( R_f = 0 \)), the boundary condition refers to the nonreflecting RCR model:  

\[
L_2 = -\frac{1}{2cZ_0} \frac{dp_{wk}}{dt}
\]  

(19)

Equation (15) is thus the general formalism of characteristic boundary conditions for R, CR, and RCR Windkessel outflow models.

E. 35-Artery Network Model

One-dimensional model of the systemic arterial tree has been proved and validated the applicability in the human circulation simulation [3], [14]. A human systemic arterial network consisting of the largest 35 arteries, shown in Fig. 2, was employed in this work for hemodynamic simulations.

Fig. 2. Arterial network model with the largest 35 systemic arteries. The arterial branches and their properties are given in Table I and Table II.

<table>
<thead>
<tr>
<th>No. ((i))</th>
<th>Artery</th>
<th>Length (L) ((\text{cm}))</th>
<th>Radius (r_o) ((\text{cm}))</th>
<th>Wave speed (c_0) ((\text{cm/s}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ascending aorta</td>
<td>4</td>
<td>1.45</td>
<td>620</td>
</tr>
<tr>
<td>2</td>
<td>Aortic arch I</td>
<td>2</td>
<td>1.12</td>
<td>580</td>
</tr>
<tr>
<td>3</td>
<td>Brachiocephalic</td>
<td>3.4</td>
<td>0.62</td>
<td>630</td>
</tr>
<tr>
<td>4</td>
<td>R. common carotid</td>
<td>17.7</td>
<td>0.37</td>
<td>680</td>
</tr>
</tbody>
</table>
### TABLE II. PERIPHERAL RESISTANCE AND COMPLIANCE DATA (MODIFIED FROM ALASTRUEY [15]).

<table>
<thead>
<tr>
<th>No. ( (i) )</th>
<th>Resistance ( R_i^c ) (mmHg s/ml)</th>
<th>Compliance ( C_i^c ) (ml/mmHg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.213E+01</td>
<td>9.600E-02</td>
</tr>
<tr>
<td>6</td>
<td>3.975E+01</td>
<td>1.280E-01</td>
</tr>
<tr>
<td>8</td>
<td>6.323E+02</td>
<td>8.000E-03</td>
</tr>
<tr>
<td>9</td>
<td>3.975E+01</td>
<td>1.280E-01</td>
</tr>
<tr>
<td>11</td>
<td>5.213E+01</td>
<td>9.600E-02</td>
</tr>
<tr>
<td>14</td>
<td>3.975E+01</td>
<td>1.280E-01</td>
</tr>
<tr>
<td>16</td>
<td>6.323E+02</td>
<td>8.000E-03</td>
</tr>
<tr>
<td>17</td>
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<td>1.280E-01</td>
</tr>
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<td>6.000E-01</td>
</tr>
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<td>7.240E-01</td>
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<td>24</td>
<td>8.250E+00</td>
<td>5.960E-01</td>
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<tr>
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<td>9.733E-02</td>
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<td>1.413E-01</td>
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<tr>
<td>33</td>
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<tr>
<td>35</td>
<td>4.200E+01</td>
<td>1.200E-01</td>
</tr>
</tbody>
</table>

The geometrical and elastic properties of the 35-artery network model are given in Table I, and the peripheral resistance and compliance data, modified from Alastruey [15], are listed in Table II. These data are the statistic average values surveyed from healthy populations, and thus they represent a generic model to be used for method development.

For hemodynamics simulations of the generic 35-artery network model, the periodic inflow boundary condition with a cardiac period of 0.8 s was imposed at the inlet of the aortic root (segment 1, Fig. 3). Each arterial segment is governed by the 1-D flow conservation laws (Eqs. 1 and 2) in conjunction with interface condition of mass conservation and pressure continuity imposed at bifurcations [13]. For the outflow boundary condition, all the 1-D terminal ends are...
coupled with the 0-D outflow models. Numerical solutions for the arterial network model were obtained by using the validated high-resolution scheme, which has been published previously [13].

Fig. 3. Periodic inflow profile.

III. RESULTS AND DISCUSSION

A. Simulation Scenarios

Usage of the present boundary condition defined in terms of the spatial derivatives \( \phi \), as given by Eq. (8), is to be exemplified. The purpose is to show that various types of 0-D outflow models can be properly simulated with relation to actual physiological situations. Numerical simulations including R, CR, and RCR outflow models are present. Besides, different sets of reflection coefficients, defined in Eq. (16), are also used for simulation.

B. Results and Discussion

The simulation takes about 10 heartbeats to achieve converged periodic waveforms. Once a quasi-steady state is reached, the pressure and flow rate waveforms calculated at a point located at the middle of the thoracic aorta II (No. 19) are illustrated in Figs. 4 and 5.

Fig. 4. Pressure waveforms simulation at a point located at the middle of the thoracic aorta II (No. 19) coupled to the following 0-D outflow models: three-element windkessel (RCR) with \( R_f=0, 0.2 \) and \(-0.2\), two-element Windkessel (CR), and single resistance (R) with \( R_f=0 \).

Fig. 5. Flow rate waveforms simulation at a point located at the middle of the thoracic aorta II (No. 19) coupled to the following 0-D outflow models: three-element windkessel (RCR) with \( R_f=0, 0.2 \) and \(-0.2\), two-element Windkessel (CR), and single resistance (R) with \( R_f=0 \).

If the 1-D terminal ends are coupled with the RCR outflow model, the system is able to capture the physiological features, such as the exponential diastolic decay and the dicrotic notch appearing in the pressure curve. In the case of \( R_f=0 \) the systolic/diastolic pressure...
is 120/69 mmHg. The heart rate was set at 75 beats/min and the calculated stroke volume was 84.9 ml, yielding a cardiac output of 6.4 l/min. These hemodynamic characteristics and features are clinically similar to what actually have been observed.

The choices of $R_f=0.2$ represents a partially reflecting end with the same sign (closed-end-like) and causes the result of blood pressure increase. In contrast, $R_f=-0.2$ represents a partially reflecting end with opposite sign (open-end-like) and causes the result of blood pressure decrease.

In addition, if the CR outflow model is prescribed at the 1-D terminal ends, it produces non-physiological oscillations either in pressure or flow waveforms due to the open-end-like fully reflecting boundary condition ($R_f=1$) [13]. Incorrect reflected waves generate from the terminal ends. The same results can be found in the previous work [16]. Finally, the single resistance ($R$) is unable to capture the physiological features of the diastolic decay rate because the physical condition of truncated downstream vascular bed is not well defined in the model.

IV. CONCLUSIONS

A general formalism of characteristic boundary conditions for R, CR, and RCR Windkessel outflow models has been proposed by introducing a parameter, termed reflection coefficients, to generalize the required numerical boundary condition treatments. A positive partial reflection coefficient ($R_f>0$) represents a boundary more closely related to closed-end nature, while a negative $R_f$ value pertains to a boundary more similar to open-end nature. The results suggest that for normal physiological simulation, terminal boundary conditions of the truncated downstream vascular bed should be simulated using RCR outflow model.

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