SWITCHING ADAPTIVE CONTROLLER FOR REGULATION OF A CLASS OF UNCERTAIN CASCADE NONLINEAR SYSTEMS

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ABSTRACT: This paper investigates the problem of global regulation for a class of cascade nonlinear systems with lower triangular or upper triangular nonlinearities. In most of the papers, usually just one of the mentioned triangular systems are considered where the nonlinearity structure is also known a priori. In this paper, the nonlinearity structure is assumed to be unknown and a switching scheme is proposed to deal with this problem. An adaptive state feedback controller which is coupled with on-line switching method is used for tuning of gain dynamics. Finally it is proved that the states of the closed loop system are globally bounded and asymptotically stable. Furthermore to show the performance of the method with simulations, the response of the system has been checked to deal with bounded delayed systems.

Keywords: Global regulation, Switching adaptive control, Unknown structure, Unknown control coefficient

I. INTRODUCTION

In the last decades, the control problem is widely investigated for nonlinear triangular systems. Many references have been obtained for nonlinear systems in the forms of the feedback or feedforward systems. Most of these papers have considered the nonlinear systems where the nonlinearities and control coefficients are assumed to be known and the problem becomes much more challenging and difficult when these terms are unknown. When the control coefficients are uncertain or unknown, some results have been obtained in [1-2]. The system considered in [2], has uncertain control coefficient with known lower and upper bounds, on the other hand global stabilization problem was investigated for feedforward systems with nonlinearities which is allowed to be lower order growing. In [2] the considered feedforward systems have unknown control coefficient with known sign and also in [3] unknown control coefficient was proposed with known sign and known lower and upper bounds for triangular systems. When uncertainties belong to only feedforward condition, state feedback control method introduced in [4] is applicable and some low gain controllers are often suggested [5]. For the cases that nonlinearities belong to only lower triangular condition, some output feedback controllers and high gain feedback control approaches are proposed [6]. Feedforward or feedback structure is a common feature of the researches in [4, 7], however by considering both classes of triangular nonlinearities, the unified feedback controller is designed to stabilize the system in [5,8].

In this paper, the extension of the method which is used in [8] is proposed for the global regulation of uncertain general cascade nonlinear systems with zero dynamics. The proposed controller is performed in the presence of the unknown nonlinearity growth rate, unknown nonlinear structure, unknown control coefficient, unknown upper and lower bounds of ISS Lyapunov function and upper bound of it’s derivatives. The proposed controller is a switching adaptive scheme which switches between high gain and low gain adaptive controllers. Particularly, this method is also applicable for either lower triangular or upper triangular nonlinear systems such as systems in [5, 8, 9] where the structure of nonlinear systems is assumed to be known and limited. In [5, 9] the nonlinear structure is assumed to be known without any switching structure.

II. SYSTEM FORMULATION

A class of uncertain cascade nonlinear system is considered:

\[
\begin{align*}
\dot{\ell} &= f_i(\ell, x), \quad i = 1, \ldots, m, \\
\dot{x}_i &= x_{i+1} + \varphi_i(t, \ell, x, u), \quad i = 1, \ldots, n - 1, \\
\dot{x}_n &= g u + \varphi_n(t, \ell, x, u).
\end{align*}
\]

where \( \ell = [\ell_1^T, \ldots, \ell_m^T]^T \in R^{n_1 + \cdots + n_m} = R^{n_1} \cdot 1 \leq m \leq n - 1 \) and \( x = [x_1, \ldots, x_n]^T \in R^n \), are the system states with initial values \( \ell(0) = \ell_0 \) and \( x(0) = x_0; u \in R \) is the control input. The control coefficient \( g \neq 0 \) is an unknown constant but the sign of \( g \), i.e., the control direction, is known. Functions \( f_i : R^{n_i} \times R^n \rightarrow R^{n_i}, i = 1, \ldots, m \) and \( \varphi_i : R \times R^{n_i} \times R^n \rightarrow R, i = 1, \ldots, n \), are locally uniformly continuous with respect to their arguments, respectively. Suppose that system (1) satisfies both following assumptions 1 and 2 for each of the cases \( (B_1) \) and \( (B_2) \).
Assumption 1: The subsystems $\zeta_i$, $i = 1, \ldots, m$ are input-to-state stable (ISS) with ISS Lyapunov function $U_i(\zeta_i)$ satisfying:

$$\varepsilon \| \dot{\zeta}_i \|^2 \leq U_i(\zeta_i) \leq \bar{c}_i \| \zeta_i \|^2, \quad \forall \zeta_i \in R^{n_i},$$

$$\dot{U}_i \leq -\alpha_i \| \zeta_i \|^2 + c_0 (x_1)^2, \quad i = 1, \ldots, m,$$

$$\varepsilon \| \dot{\zeta}_i \|^2 \leq U_i(\zeta_i) \leq \bar{c}_i \| \zeta_i \|^2, \quad \forall \zeta_i \in R^{n_i},$$

$$\dot{U}_i \leq -\alpha_i \| \zeta_i \|^2 + c_0 \sum_{j=i+2}^{n} x_j^2, \quad i = 1, \ldots, m,$$

where $\varepsilon, \bar{c}_i, \alpha_i, i = 1, \ldots, m$ and $c_0$ are unknown positive constants ($\alpha_i \geq 1$).

Assumption 2: There exist an unknown constant $\mathcal{C} > 0$ such that:

$$|\varphi_i(t, \zeta, x, u)| \leq \mathcal{C} \left( \sum_{j=1}^{m} \| \zeta_j \| + \sum_{j=1}^{i} |x_j| \right), \quad i = 1, \ldots, n,$$

$$|\varphi_i(t, \zeta, x, u)| \leq \mathcal{C} \left( \sum_{j=1}^{m} \| \zeta_j \| + \sum_{j=i+2}^{n} |x_j| \right), \quad i = 1, \ldots, n, \quad n \geq 3.$$

Note that $(B_1)$ is often known as a lower triangular (strict feedback) condition [6], whereas $(B_2)$ is often known as an upper triangular (feedforward) condition [1, 9]. Similar and more general assumptions are given in [7].

In [8] it is taken a further extension so that two dynamic gains are designed, one for triangular and the other for feedforward form. In fact the method in [8] is proposed for developing the individual adaptive controller using introduced method in [9]. The class of nonlinear systems considered in [11]

III. MAIN RESULT

The proposed feedback controller is given by

$$u = K(\varepsilon(t))x,$$

where

$$K(\varepsilon(t)) = \begin{bmatrix} k_1/\varepsilon(t)^{n_1}, & k_2/\varepsilon(t)^{n_2}, & \ldots, & k_n/\varepsilon(t)^{n_n} \end{bmatrix}, \varepsilon(t) > 0.$$  

The dynamic gain $\varepsilon(t)$ will be introduced later. Throughout this paper: $I_n$ denotes the $n \times n$ identity matrix,

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g \end{bmatrix},$$

other notations are as follows:

$$E_{\varepsilon(t)} = \text{diag}[1, \varepsilon(t), \ldots, \varepsilon(t)^{n-1}], \quad K = K(1), \quad A_K = A + BK, \quad \text{and} \quad A_{\varepsilon(t)} = A + BK(\varepsilon(t)).$$
Using (1) and (6), following closed loop system can be achieved
\[
\begin{align*}
\dot{\zeta}_i &= f_i, i = 1, \ldots, m, \\
\dot{x} &= A_K(\epsilon(t))x + \varphi(t, \zeta, x, u).
\end{align*}
\] (7)

Lemma 1: Let \( V_\zeta(x) = x^TP_\zeta(\epsilon(t))x + R(\epsilon(t))\sum_{i=1}^{m} U_i(\zeta) \) where \( P_\zeta(\epsilon(t)) = E_\epsilon P_\zeta E_\epsilon(t) \).

From reference \([5]\), \( A_K(\epsilon) = \epsilon^{-1}E_\epsilon^{-1}A_K E_\epsilon \) and the Lyapunov equation \( A_K(\epsilon(t))^TP_\zeta(\epsilon(t)) + P_\zeta(\epsilon(t))^TA_K(\epsilon(t)) = -\epsilon(t)^{-1}E_\epsilon(t)E_\epsilon(t)^T \) and \( A_K^TP_\zeta + P_\zeta A_K = -I \). Along the state trajectory of (7) and according to [8], following inequality can be achieved
\[
V(\zeta_x) \leq - (\epsilon(t)^{-1} - \sigma_1)\|\epsilon(t)\|E_{\epsilon(t)}X^T ||E_{\epsilon(t)}X^T \|^2 + \sigma_2\|E_{\epsilon(t)}x\|\|E_{\epsilon(t)}\varphi(t, \zeta, x, u)\|_1 + \sum_{i=1}^{m} \zeta_i \|_2^2 + R(\epsilon) \sum_{i=1}^{m} -\alpha_i \|_2^2 + c_0 m \|x\|_2^2 \] (8).

where \( \sigma_1 = 2s_n\|P_\zeta\|, \sigma_2 = 2\|P_\zeta\|, s_n = \sqrt{n(n-1)(2n-1)/6} \). Let’s consider the following two gain dynamics \( \epsilon_1(t) \) and \( \epsilon_2(t) \) where \( \epsilon_1(t) \) is monotonically non-increasing and \( \epsilon_2(t) \) is monotonically non-decreasing
\[
\begin{align*}
\epsilon_1(t) &= -\frac{\|x\|_1}{\|x\|_1 + 1} \epsilon_1^{n+1}(t), \epsilon_1(t_0) = 1, \\
\epsilon_2(t) &= \frac{\|x\|_1}{\|x\|_1 + 1} \epsilon_2^{-1}(t), \epsilon_2(t_0) = 1.
\end{align*}
\] (9) (10)

Lemma 2: Given assumptions 1 and 2 and using controller (6) with both \( \epsilon(t) = \epsilon_1(t) \) or \( \epsilon(t) = \epsilon_2(t) \), all states of the closed-loop system (1) are globally regulated if \( \lim_{t \to \infty} \epsilon(t) = \bar{\epsilon}, 0 < \bar{\epsilon} < \infty \).

Proof of Lemma 2: First we need to show that, there is no finite-time escape phenomenon in the class of nonlinear cascade systems (1). According to the definition ISS and iISS in [10], ISS implies that \( \dot{x} = f(x, u) \) is bounded-input bounded-state stable when \( u \neq 0 \) and that the zero solution (with \( u = 0 \)) is global asymptotically stable. However, the converse is not true in general. For the iISS property, it is shown that iISS is strictly weaker than ISS. Both ISS and iISS properties can be equivalently characterized using Lyapunov functions. By assumption 1, the \( \zeta \)-subsystem is ISS, so the states are limited and therefore finite-time escape phenomenon cannot appear in this subsystem. Given (4)-(5) nonlinear functions \( \varphi(t, \zeta, x, u) \) do not grow too fast, therefore the finite time escape phenomenon does not exist. To further emphasize, the variant of Lemma 3 in [11] is provided, this Phenomenon occurs in the following conditions. Consider the system:
\[
\begin{align*}
\dot{X}_1 &= f_1(t, X_1, X_2) \\
\dot{X}_2 &= f_2(t, X_1, X_2).
\end{align*}
\] (11)
where \( X_1, X_2 \in \mathbb{R} \), and \( f_1, f_2 \in C^1 \). If there exists a nonempty open set \( U \subset \mathbb{R} \), and positive constants, \( a, b, c, d, s \) and \( \vartheta > 2, r > \vartheta - 1 \) and \( a > b \) such that, for all \( t \in [0, s], X_1 \subset U \) and \( X_2 > d \),
\[
|f_1(t, X_1, X_2)| < cX_1^r, \quad bX_2^\vartheta < f_2(t, X_1, X_2) < aX_2^\vartheta.
\] (12)
then, for \( X_1(0) \in U \) and sufficiently large \( X_2(0) > d, X_2(t) \) escapes to infinity in finite time.

Considering, \( \dot{\bar{x}}_i = x_{i+1} + \varphi_i(t, \zeta, x, u) \)

let’s define \( X_1 = [x_1, x_2, \ldots, x_{n-1}]^T, X_2 = x_n, f_1 = [x_2 + \varphi_1(t, \zeta, x, u), \ldots, x_n + \varphi_{n-1}(t, \zeta, x, u)]^T \) and \( f_2 = gu + \varphi_n(t, \zeta, x, u) \), if the assumption (12) satisfy for \( f_1 \) and \( f_2 \), then in \( x_n = X_2 \) finite time escape phenomenon occurs. Note that above assumption does not satisfy for \( f_2 \) and so finite time escape phenomenon does not occur.

(i) Case \( \epsilon(t) = \epsilon_1(t) \): let’s consider following closed loop solution for the \( \epsilon_1(t) \)
\[
\epsilon_1(t) = \frac{1}{\epsilon_1(t_0)^n} + \int_{t_0}^{t} \frac{\|x(s)\|_1}{\|x(s)\|_1 + 1} ds \leq \frac{1}{\epsilon_1(t_0)^n} \] (13)

given (13), \( \lim_{t \to \infty} \epsilon(t) = \bar{\epsilon} \) implies that \( \lim_{t \to \infty} \int_{t_0}^{t} \frac{\|x(s)\|_1}{\|x(s)\|_1 + 1} ds \) exist and is finite, since \( \frac{\|x\|_1}{\|x\|_1 + 1} \) is uniformly continuous on \([t_0, \infty)\), based on Barbalat’s lemma [20], it is obvious that \( \|x\|_1 \to 0 \) as \( t \to \infty \). Integrating (8), following inequality for sub states \( \zeta \) can be achieved as
\[
\sum_{i = 1}^{m} \int_{t_0}^{\infty} \| \dot{z}_i(t) \|^2 \leq \frac{1}{\gamma_i} \left\{ \nu(\bar{z}(0), x(0)) + \int_{t_0}^{\infty} \left( \sigma_1 \| \dot{z}_1(t) \| \| \dot{z}_1(t) \|^{-1} \right) E_{t_0}(\dot{z}_1(t)) x(t) \| \| E_{t_0}(\dot{z}_1(t)) x(t) \| \| x(t) \|^2 \right. \\
+ \sigma_2 \| E_{t_0}(\dot{z}_1(t)) \| \| E_{t_0}(\dot{z}_1(t)) \| \| x(t) \|^2 \left\| \varphi(t, \bar{z}, x, u) \| \right. \\
+ R(\bar{z}_i) c_0 m \| x(t) \|^2 \right\} dt \right. \\
\]

(14)

where \( \gamma_i = R(\bar{z}) \alpha_i - |\bar{R}(\bar{z})| \bar{e}_i \). Since \( \| x(t) \| \rightarrow 0 \) as \( t \rightarrow \infty \), then \( \| x(t) \| \rightarrow 0 \) and according to the fact that \( \| \bar{e}_i(t) \| \) is uniformly continuous, \( \| x(t) \| \) is also uniformly continuous, thus using Barbalat’s lemma, \( \int_{t_0}^{\infty} \| x(t) \|^2 dt \) converges. The inequality \( \int_{t_0}^{\infty} \sigma_1 \| \dot{z}_1(t) \| \| \dot{z}_1(t) \|^{-1} \| E_{t_0}(\dot{z}_1(t)) x(t) \| \| x(t) \| \| x(t) \|^2 dt \leq \max(\sigma_1 \| \dot{z}_1(t) \| \| \dot{z}_1(t) \|^{-1} \| E_{t_0}(\dot{z}_1(t)) \| \| x(t) \| \| x(t) \|^2) \int_{t_0}^{\infty} \| x(t) \|^2 dt \)

and

\[
\int_{t_0}^{\infty} R(\bar{z}_i) c_0 m \| x(t) \|^2 dt \leq \max(\sigma_1 \| \dot{z}_1(t) \| \| \dot{z}_1(t) \|^{-1} \| E_{t_0}(\dot{z}_1(t)) \| \| x(t) \| \| x(t) \|^2) \int_{t_0}^{\infty} \| x(t) \|^2 dt
\]

is bounded and given \( \| x(t) \| \| \varphi(t, \bar{z}, x, u) \| _1 \) is uniformly continuous and \( \| x(t) \| \| \varphi(t, \bar{z}, x, u) \| _1 \rightarrow 0 \) as \( t \rightarrow \infty \), so

\[
\int_{t_0}^{\infty} \sigma_2 \| E_{t_0}(\dot{z}_1(t)) \| \| \varphi(t, \bar{z}, x, u) \| _1 \| x(t) \|^2 dt \leq \max(\sigma_1 \| E_{t_0}(\dot{z}_1(t)) \| \| E_{t_0}(\dot{z}_1(t)) \| \| \varphi(t, \bar{z}, x, u) \| _1 \| x(t) \|^2) \int_{t_0}^{\infty} \| x(t) \| ^2 dt
\]

is bounded using Barbalat’s lemma. Based on above dissuasions, \( \int_{t_0}^{\infty} \| \dot{z}_i(t) \|^2 dt \) converges and as \( \bar{z} \) are bounded, based on (1) it is concluded that \( \bar{z}_i \) is bounded and with boundedness of \( \bar{z}_i \), it can be concluded that \( \lim_{t \rightarrow \infty} \| \bar{z}_i \| = 0 \).

(ii) case \( \varepsilon(t) = \varepsilon_\varphi(t) \): let’s consider following closed loop solution for the \( \varepsilon_\varphi(t) \)

\[
\varepsilon_\varphi(t) = \sqrt{\frac{1}{2} \int_{t_0}^{t} \| x(s) \|^2 ds + \varepsilon_2(t_0)^2}.
\]

(15)

Similar to the case (i), it can be easily seen that \( \lim_{t \rightarrow \infty} \varepsilon_\varphi(t) = \bar{\varepsilon} \) implies that \( \| x(t) \| \rightarrow 0 \) and \( \lim_{t \rightarrow \infty} \bar{z}(t) = 0 \).

Lemma 3: suppose that either \( B_1 \) or \( B_2 \) holds. Select \( \varepsilon(t) = \varepsilon_1(t) \) and \( \varepsilon(t) = \varepsilon_2(t) \) under \( B_2 \), the controller (6) is selected with \( \varepsilon(t) = \varepsilon_1(t) \) and

Next, we introduce a switching adaptive scheme such that the controller (6) can regulate all states of the closed loop system (1) with two dynamic gains (9) and (10) and a switching logic as follows:

Initialization:

Set two dynamic gains \( \dot{\varepsilon}_1 = -\frac{\| x(t) \|}{\| \bar{e}_i(t) \|} \varepsilon_1(t) \) and \( \dot{\varepsilon}_2 = \frac{\| x(t) \|}{\| \bar{e}_i(t) \|} \varepsilon_2(t) \), \( \dot{\varepsilon}_1(t_0) = 1 \), \( \dot{\varepsilon}_2(t_0) = 1 \).

Define \( \varepsilon_1(t) = \varepsilon_1(t_0) \) and \( \varepsilon_2(t) = \varepsilon_2(t_0) \) for \( t_1 < t < t_{i+1} \) where \( \varepsilon_1(t) = \varepsilon_1(t_0) + \| E \varepsilon_1(t) x(t) \| \) and \( \varepsilon_2(t) = \varepsilon_2(t_0) + \| E \varepsilon_2(t) x(t) \| \)

Set either \( \varphi_0 = 1 \) or \( \varphi_0 = 2 \).

Switching logic:

Step 1: Set \( \varepsilon(t) = \varepsilon_\varphi(t) \) for \( t_1 < t < t_{i+1} \).

Step 2: If

\[
d\int_{t_0}^{t} \| E \varphi(s) x(s) \| ds > (i + 1) \theta \varphi(t).
\]

(16)

Go to step 3 otherwise, go to step 2.

Step 3: Set \( t_{i+1} = t \), \( \varphi_{i+1} \in H \{ \varphi_i \} \) where \( H = \{1, 2\} \). Let \( i = i + 1 \), go to step 1.

Note that the dynamic gains, states and switching logic are calculated at any point in time.

We briefly give an intuition about how the switching logic is designed and how it works. First two dynamic gains \( \varepsilon_1(t) \) for \( B_1 \) and \( \varepsilon_2(t) \) for \( B_2 \) is designed. It is deduced that the integral function (16) has a finite value, if \( \varepsilon_1(t) \) becomes sufficiently small for \( B_1 \) and \( \varepsilon_2(t) \) becomes sufficiently large for \( B_2 \). Since nonlinearity structure is not known, two dynamic gains \( \varepsilon_1(t) \) and \( \varepsilon_2(t) \) are switched alternatively by monitoring the integral function (16). Eventually, the integral function (16) has a finite value, also the switching stops in a finite time, and \( \varepsilon(t) \) converges to a finite value [8]. The switching occurs by using the monitoring signal generator. The condition (16) is
conceptually similar with the monitoring signal generator in [9].

**Theorem 1:** Suppose that assumptions 1 and 2 hold. Select $K$ such that $A_K$ is Hurwitz. Then with the controller (6) and switching logic, $\lim_{t \to \infty} \epsilon(t) = \tilde{e}$, $0 < \tilde{e} < \infty$ and all states of the closed loop system (1) are globally regulated.

**Proof of theorem 1:** We consider two parts for proof. Note that there is no finite escape phenomenon with the proposed controller. In part 1, the finite number of switching in dynamic gain will be proved and in part 2, it is shown that closed loop system is regulated in finite switching.

Part 1: In this part we prove that the dynamic gain is switched only in finite number of times. Consider two cases: case (I): the system nonlinearity $\phi(t, \tilde{z}, x, u)$ actually satisfies (B_1), but it is not known a priori; case (II): the system nonlinearity $\phi(t, \tilde{z}, x, u)$ actually satisfies (B_2), but it is not known a priori.

Case (I): We will show that switching stops by proving that $\tilde{e}_1(t)$ can only be selected in finite times. Let $l_i$ denotes the switching index when $\tilde{e}_1(t)$ is selected for the first time. Thus $l_i$ is either 0 or 1. Define $V_i(t, x) = x^T P_i (\tilde{e}_1(t)) x + \tilde{e}_1^{2(n-1)} \sum_{i=1}^m U_i (R (\tilde{e}_1 (t))) = R (\tilde{e}_1 (t)) = \tilde{e}_1^{2(n-1)}$ as a Lyapunov function for $t \in [t_k, t_{k+1})$ where $k \in H_1$ is defined by $H_1 = \{k| k = l_i + 2(j-1), j = 1,2, \ldots \}$. Also, for $t \in [t_k, t_{k+1})$, we have

$$\tilde{e}_1^{2(n-1)} \sum_{i=1}^m U_i \| \tilde{e}_1 \|^2 + \lambda_1 \| E_{\tilde{e}_1} x \|^2 \leq V_i(t, x) \leq \lambda_2 \| E_{\tilde{e}_1} x \|^2 + \tilde{e}_1^{2(n-1)} \sum_{i=1}^m U_i \| \tilde{e}_1 \|^2. \quad (17)$$

where $\lambda_1 = \lambda_{\min} (P_i)$, $\lambda_2 = \lambda_{\max} (P_i)$. Then, along the trajectory of (7) and using the technique of the completion of squares (for $a, b \in \mathbb{R}$ and $\Delta > 0$ the inequality $2ab \leq \frac{1}{\Delta} a^2 + b^2$ holds):

$$\dot{V}_i(t, x) \leq -(\tilde{e}_1(t)) - \alpha_1 \epsilon_1(t) \tilde{e}_1(t)^{-1} - \alpha_2 (1 + \tilde{e}_1^{2(n-1)} \sum_{i=1}^m U_i \| \tilde{e}_1 \|^2. \quad (18)$$

where $0 < \alpha_1, \alpha_2 < 1$. Under (B_1), note that $\| E_{\tilde{e}_1} \phi(t, \tilde{z}, x, u) \| \leq M_1 (\tilde{e}_1(t)) (\| E_{\tilde{e}_1} x \| + \frac{1}{\sqrt{n}} \sum_{i=1}^m \| \tilde{e}_1 \|)$ where $M_1 (\tilde{e}_1) = C \sqrt{n} (1 + \tilde{e}_1(t) + \tilde{e}_1(t)^2 + \epsilon_1(t)^{n-1})$ and $\beta_{il} = -\frac{1}{2k_i} \tilde{e}_1^{2(n-1)} + \tilde{e}_1^{2(n-1)} \alpha_1 - 2(n-1) \epsilon_1(t) \tilde{e}_1(t)^{2n-3} \tilde{e}_i$

using (18), we have:

$$\dot{V}(t, x) \leq -(1-c) \epsilon_1(t)^{-1} \| E_{\tilde{e}_1} x \|^2 - T (\epsilon_1(t)) \| E_{\tilde{e}_1} x \|^2 - \sum_{i=1}^m \beta_{il} \| \tilde{e}_i \|^2. \quad (19)$$

for $t \in [t_k, t_{k+1})$ where, $T(e_1(t)) = c' \epsilon_1(t)^{-1} - \sigma_1 \epsilon_1(t) \epsilon_1(t)^{-1} - \sigma_2 (1 + \tilde{e}_1^{2(n-1)} \sum_{i=1}^m U_i \| \tilde{e}_1 \|^2. \quad (19)$

Recall that $\epsilon_1(t)$ is monotonically non-increasing. So it is clear that there exists $\epsilon_1^*$ such that $\epsilon_1(t) < \epsilon_1^*$ to satisfy $T (\epsilon_1(t)) > 0$ and $\beta_{il} > 0$. Using Lemma 2 if $\epsilon_1(t)$ converges to a value larger than or equal to $\epsilon_1^*$, then the closed loop system (1) is trivially regulated. Thus we only need to consider a case that $\epsilon_1(t) < \epsilon_1^*$. Let $\tilde{e}_1(t)$ be the smallest switching index such that $\epsilon_1(t) < \epsilon_1^*$ for $t > t_i$. As mentioned $\epsilon_1(t)$ is monotonically non-increasing and with power $2(n-1)$ it is very small and with selection $\tilde{e}_1, \| \tilde{e}_1 \| sufficiently small, we can ignore this term. Then from (17), (19), it is obtain, for $t \in [t_i, t_{i+1})$ where $i \in H_1$ defined by $H_1 = \{i| i = k + 2(j-1), j = 1,2, \ldots \} \subset H_1$

$$\| E_{\tilde{e}_1} x \| \leq \frac{\lambda_2}{\lambda_1} \| E_{\tilde{e}_1} x(t_i) \| e^{-(1-c') \epsilon_1(t_i)^{-1} \frac{t}{t_i}}. \quad (20)$$

let $t_i \in H_1$ is the smallest integer that satisfies:

$$i \geq \frac{2 \lambda_2}{(1-c') \sqrt{\lambda_1}} \frac{1}{\lambda_2} \quad (21)$$

since $\epsilon_1(t)$ is monotonically non-increasing, $\int_{t_i}^{t} \epsilon_1(t)^{-1} ds \geq \epsilon_1(t_i)^{-1}(t - t_i)$. After that the $i_t$ th switching time accurate, using (20):

$$\int_{t_i}^{t} \| E_{\tilde{e}_1} x(s) \| ds = \int_{t_i}^{t} \frac{\lambda_2}{\lambda_1} \| E_{\tilde{e}_1} x(t_i) \| e^{-(1-c') \epsilon_1(t_i)^{-1} (t-t_i)} ds \leq \frac{2 \lambda_2}{(1-c') \sqrt{\lambda_1}} \frac{1}{\lambda_2} \| E_{\tilde{e}_1} x(t_i) \| \| E_{\tilde{e}_1} x(t_i) \| \quad (22)$$
from (21) and (22), we have:
\[
\int_{t_i}^{t} \|E_{\varepsilon(t)}x(s)\| \, ds \leq (i + 1) \theta_1(t). 
\]
from (23), no further switching occurs for \( t > t_i \).

Case (II): We will show that switching stops by proving that \( \varepsilon_2(t) \) can only be selected finitely in times. Let \( l_2 \) denotes the switching index when \( \varepsilon_2(t) \) is selected for the first time. Thus \( l_2 \) is either 0 or 1. Define \( V_2(\lambda, x) = x^T P_k \varepsilon_2(t) x + \varepsilon_2^{-2(n-1)} \sum_{i=1}^{m} U_i R(\varepsilon_2(t)) = R_2(\varepsilon_2(t)) = \varepsilon_2^{-2(n-1)} \) as a Lyapunov function for \( t \in [t_k, t_{k+1}] \) where \( k \in H_2 \) defined by \( H_2 = \{ k | k = l_2 + 2(j-1), j = 1, 2, \ldots \} \). Also, for \( t \in [t_k, t_{k+1}] \), we have:
\[
\varepsilon_2^{-2(n-1)} \sum_{i=1}^{m} U_i l_i^2 + \lambda_1 \|E_{\varepsilon_2(x)}\|^2 \leq V_2(\lambda, x) 
\]
\[
\leq \lambda_2 \|E_{\varepsilon_2(x)}\|^2 + \varepsilon_2^{-2(n-1)} \sum_{i=1}^{m} \xi_i l_i^2. 
\]
(24)
then, along the trajectory of (7) and also using mentioned equation 2ab \( \leq \frac{1}{\Delta} \alpha^2 + \Delta b^2 \):
\[
V_2(\lambda, x) \leq -\varepsilon_2(t)^{-1}(1 - \sigma_1 \varepsilon_2(t) - \sigma_2 (\varepsilon_2 + \varepsilon_2^{2(n+1)} m_0 \xi \varepsilon_2^{-2(n-1)})) M_2(\varepsilon) - \varepsilon_2^{-2(n+3)}|c_0| \|E_{\varepsilon_2(x)}\|^2 - \sum_{i=1}^{m} \beta_2 l_i^2. 
\]
(25)
where \( 0 < k_2 < 1 \). Under \((B_2)\). Note that \( \|E_{\varepsilon_2(x)}\|^2 \leq M_2(\varepsilon) \|E_{\varepsilon_2(x)}\|^2 + e_n \sum_{i=1}^{m} l_i^2 \), where \( M_2(\varepsilon) = e_n(\varepsilon_2(t)^{-1} + \varepsilon_2(t)^{-2} + \cdots + \varepsilon_2(t)^{-m}) \) and \( \beta_{2i} = -\frac{1}{e_2} \varepsilon_2^{-2(n-1)} + \varepsilon_2^{-2(n-2)} \alpha_i + 2n(n-1) \varepsilon_2(t)^{-2n+1} \), \( n \geq 3 \).

using (25), we have:
\[
V_2(\lambda, x) \leq -(1 - c^t) \varepsilon_2(t)^{-1} \|E_{\varepsilon_2(x)}\|^2 
\]
\[
- T(\varepsilon_2(t)) \|E_{\varepsilon_2(x)}\|^2 
\]
\[
- \sum_{i=1}^{m} \beta_2 l_i^2. 
\]
(26)
for \( t \in [t_k, t_{k+1}] \) where \( T(\varepsilon_2(t)) = \varepsilon_2(t)^{-1}(c' - \sigma_1 \varepsilon_2(t)) - \sigma_2 (\varepsilon_2 + \varepsilon_2^{2(n+1)} m_0 \xi \varepsilon_2^{-2(n-1)}) M_2(\varepsilon) - \varepsilon_2^{-2(n+3)} m_0 \) and \( 0 < c' < 1 \). Given that \( \varepsilon_2(t) \) is monotonically non-decreasing, it is clear that there exists \( \varepsilon_2^* \) such as \( \varepsilon_2(t) > \varepsilon_2^* \) to satisfy \( T(\varepsilon_2(t)) > 0 \) and \( \beta_{2i} > 0 \). Using Lemma 2 if \( \varepsilon_2(t) \) converges to a value less than or equal to \( \varepsilon_2^* \), the closed loop system (1) is trivially regulated. Thus we only need to consider the case \( \varepsilon_2(t) > \varepsilon_2^* \). Let \( i_2 \in H_2 \) is the smallest switching index such that \( \varepsilon_2(t) > \varepsilon_2^* \) for \( t > t_i \). As mentioned \( \varepsilon_2(t) \) is monotonically non-decreasing and with power \( -2(n-1) \) it is very small and with selection \( \xi_i \xi_i \) sufficiently small, we can ignore this term. Then from (24) and (26) it is easy to obtain, for \( t \in [t_i, t_{i+1}) \) where \( i \in H_2 \) defined by \( H_2 = \{ i | i = i_2 + 2(j-1), j = 1, 2, \ldots \} \),
\[
\|E_{\varepsilon_2(t)}x\|^2 \leq \left( \frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{2}} \|E_{\varepsilon_2(t_i)}x(t_i)\| e^{-\frac{(1-c^t)}{2(\varepsilon_2^*)}} \|E_{\varepsilon_2(x)}\|^2. 
\]
(27)
note that \( \varepsilon_2(t) = \min \left\{ 2 \int_{t_i}^{t} \|x\|^2 + \varepsilon_2(t_i), e_n \right\} \), and given \( \int_{t_i}^{t} \|x\|^2 \, ds \leq t - t_i \),
\[
\varepsilon_2(t) \leq \sqrt{2(t - t_i) + \varepsilon_2(t_i)}^2. 
\]
(28)
then from (28):
\[
\frac{1}{e_n} \sum_{i=1}^{m} l_i^2 \geq \sqrt{2(t - t_i) + \varepsilon_2(t_i)}^2 - \varepsilon_2(t_i). 
\]
(29)
let’s consider \( i_f \in H_2 \) as a smallest integer that satisfying:
\[
i_f \geq \frac{2 \lambda_2}{(1-c^t) e_n} \frac{1}{\sqrt{\lambda_1}} \left( 1 + \frac{2 \lambda_2}{(1-c^t) e_n} \right). 
\]
(30)
after that the \( i_f \) th switching time accurate, using (27) and (29) and \( \varepsilon_2(t_{i_f}) \geq 1 \)
\[
\int_{t_{i_f}}^{t} \|E_{\varepsilon_2(x)}\|^2 \, ds \leq 2 \lambda_2 \left( \frac{1}{1-c^t} \right) \frac{1}{\sqrt{\lambda_1}} \left[ \left( 1 + \frac{2 \lambda_2}{(1-c^t) e_n} \right) \|E_{\varepsilon_2(t_{i_f})}x(t_{i_f})\| 
\]
(31)
from (30) and (31), we have:
\[
\int_{t_{i_f}}^{t} \|E_{\varepsilon_2(x)}\|^2 \, ds \leq (i_f + 1) \theta_2(t). 
\]
(32)
from (32), no further switching occurs for \( t > t_f \).

Part 2: The closed loop system (1) is regulated for finite switching. See proof in [8].

IV. ILLUSTRATIVE EXAMPLE

Consider the following feedforward nonlinear system [1]:

\[
\begin{align*}
\dot{\ell}_4 &= -\ell_4 + |x_3|, \\
\dot{x}_1 &= x_2 + 0.5\ell_4 + 0.5x_3 \sin x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= gu.
\end{align*}
\]

\[ (33) \]

If \( n = 3 \), \( m = 1 \), suppose \( g = -2 \), \( K = [2, 4, 3] \), \( x(0) = [-5.2, -0.2]^{T} \). \( \ell_4(0) = 0.2 \). Uncertain nonlinearities satisfies assumptions 1 and 2. Figs. 1 and 2 show the results of the proposed method.

![Fig. 1. The responses of the closed-loop system, a) state trajectories, b) dynamic gain for the case \( \varphi_0 = 1 \).](image)

![Fig. 2. The responses of the closed-loop system, a) state trajectories, b) dynamic gain for the case \( \varphi_0 = 2 \).](image)

The states convergence in the proposed method for both cases \( \varphi_0 = 1 \) and \( \varphi_0 = 2 \) show that the system simultaneously regulated without a prior knowledge on their nonlinearity structure, growth rate and control coefficient. The control input and state trajectories are contained within reasonable bounds, i.e., there is no big spike at each switching point. Fig. 3 and 4 show simulation results of the system (33) with delay 0.1 seconds for the case \( x_3 \) respectively for the case \( \varphi_0 = 1 \) and \( \varphi_0 = 2 \).

V. CONCLUSION

In this paper, we proposed a switching adaptive controller for a class of nonlinear cascade uncertain systems with expanding the limited conditions on the nonlinearity for developing the method to a larger class of nonlinear systems with upper triangular or lower triangular forms. Our purpose method is to use the unified method to stabilize and regulate triangular systems, so the adaptive state controller is coupled with bi-directional switching controller. The proposed control law is online and it is calculated at each time. Moreover this method has been checked for systems with limited bounded delays which shows a good performance of the proposed approach. It is proved that using the suggested method, the nonlinear systems with unknown rate of growth condition, unknown nonlinearity structure, unknown control coefficient, unknown bounds of ISS Lyapunov function and it’s derivatives are regulated and all the states of closed loop system remain bounded.

![Fig. 3. The responses of the closed-loop system with delay 0.1 sec, a) state trajectories, b) control input for the case \( \varphi_0 = 1 \).](image)

![Fig. 4. The responses of the closed-loop system with 0.1 delay a) state trajectories, b) control input for the case \( \varphi_0 = 1 \).](image)
Similarly for $\{B_2\}$ case: since $\epsilon(t) = \epsilon_2(t)$ has a well-defined closed-form solution (15), the closed-loop system (7) has a unique solution $(\ell(t), x(t), \epsilon(t))$ on $[t_0, T_f]$ for some $T_f \in (t_0, \infty)$. For showing that $(\ell(t), x(t), \epsilon(t))$ is well-defined and bounded, according to mentioned Lyapunov function in Case (II) in the proof of theorem 1 and equation (25), since $\epsilon(t)$ is monotonically non-decreasing, it is clear that there exists $\epsilon_2^*$ such as $\epsilon(t) > \epsilon_2^*$ to satisfy $1 - \sigma_1 \epsilon_2(t) - \sigma_2 \left( 1 + \sigma_2 \frac{M_1(\epsilon_2)}{2^\gamma} k_2 \epsilon_2 \right) \frac{2}{2^\gamma} > \rho $ and $\beta_2 > 0$ and $0 < \rho < 1$. First we consider a case that $\epsilon(t)$ converges to a value less than or equal to $\epsilon_2^*$. Using lemma 2, all states of the closed loop system (1) is trivially regulated. Thus we only need to consider a case that there exists time $t_2 \in [t_0, T_f]$ such that $\epsilon(t) > \epsilon_2^*$ for $t \in [t_2, T_f]$ then we have

$$
V(\ell, x) \leq -\frac{\rho}{\epsilon(t)} \| E(\ell, x) \|^2 - \sum_{i=1}^{m} \beta_{2i} \| \ell_i \|^2. \tag{36}
$$

according to $\epsilon(t) = \epsilon_2(t)$ is monotonically non-decreasing and with ignoring mentioned terms in theorem 1 and from (38), it is deduced that $\epsilon_2(t)$ reaches a finite constant as $T_f$ becomes larger, and then by using lemma 2, $(\ell, x) \to 0$.

**REFERENCES**


