Default Contagion and Systemic Risk in the Presence of Credit Default Swaps

Katsumasa Nishide  
Graduate School of Economics,  
Hitotsubashi University  
2-1 Naka, Kunitachi,  
Tokyo 186-8601, Japan

Teruyoshi Suzuki  
Graduate School of Economics and Business Administration  
Hokkaido University  
Kita 8, Nishi 5, Kita-ku, Sapporo, Japan  
Hokkaido 060-0808 Japan

Kyoto Yagi  
Graduate School of Management,  
Tokyo Metropolitan University  
1-4-1 Marunouchi, Choyoda,  
Tokyo 100-0005, Japan

Abstract: We consider a clearing system of an interbank market in the case cross-trading of credit default swaps among banks is present, and we investigate how the cross-trading affects financial markets.

Keywords: Credit default swap, default contagion, systemic risk.

I. Introduction

We introduce cross-trading of credit default swaps (hereafter CDS) among banks into a model constructed by the seminal paper [3]. One of the main difficulties in this research is how to prove the existence of a clearing payment vector, equivalently the payoffs of financial products consistent with the cross-trading structure. Mathematically, a clearing payment vector is expressed as a fixed point, implying that the fixed point theorem should be applied for the proof. In our case where CDS cross-trades and default costs are present, the fixed point theorem cannot simply be applied because the payoff function might not be increasing nor continuous. The main results are as follows. To guarantee the existence of a clearing payment vector, we propose the fictitious default algorithm with financial covenants, which is a modification of the algorithm originally proposed by [3]. After proving the existence of a clearing payment vector under the assumption of our default algorithm, we conduct numerical calculations to analyze the effect of CDS cross-trades quantitatively. With our simple example, it is concluded that the cross-trading of CDSs among banks can have a big and negative impact on market stability.

II. The Model

Consider a one-shot economy with current and maturity times. There are $n=3$ banks in total in the financial market. We denote the business (external) asset of bank $i$ at maturity by $e_i$ and write $e = (e_1; \ldots; e_n)^T$. We assume that the realization of $e$ is independent of capital or trade structures of the banks. If bank $i$ defaults at maturity, its business asset is reduced to $(1-c_i)e_i$ when bank $i$ defaults. Let $D$ be the set of defaulting banks and define the diagonal matrix $\Delta$ by $\Delta = \text{diag}(\ell D(i)gn \ i=1)$. In other words, $\Delta$ is a diagonal matrix whose $i$-th element is equal to 1 if bank $i$ defaults. Then, the vector of banks’ business assets after the reduction of default costs is given by $(I - \Delta) e$

1Likewise, we write column vectors with bold letters as $a = (a_1; \ldots; a_n)^T$
Three types of financial securities are issued and traded by banks. The first type is a straight debt. Bank $j$ issues a debt with face value $p_j$. The second type is a CDS, whose payoff is explained later. The third type is an equity, whose payoff is positive if there are some surplus assets after paying debt and CDSs. The cross-trade structure is described as follows. Let $m_{kj}$ be bank $i$'s proportion of ownership of the CDS issued by bank $j$ with reference to bank $k$, $m_{ij}$ be bank $i$'s proportion of ownership of bank $j$'s straight debt, and $m_{ij}^0$ be bank $i$'s proportion of ownership of bank $j$'s equity. Then, the ownership structure associated with bank $k$'s debt is written as the matrix

$$M^k = \begin{pmatrix}
    m_{11}^k & \cdots & m_{1n}^k \\
    \vdots & \ddots & \vdots \\
    m_{n1}^k & \cdots & m_{nn}^k
\end{pmatrix}$$

**Assumption 1** For any $k$, matrix $M^k$ is substochastic. Next, we explain the payoff structure of CDSs. We denote by $p_{kj}$ the final payoff of the CDS written by bank $j$ with reference to bank $k$, $p_{ij}$ the payoff of the straight debt issued by bank $j$, and $p_{ij}^0$ the payoff of bank $j$'s equity. Let $d_{kj}$ be the proportion of CDSs that bank $j$ writes with reference to bank $k$'s debt in the interbank market. We can naturally set $d_{kj} = 1$. In other words, if bank $j$ does not default, the payoff of CDS written by bank $j$ with reference to bank $k$ is given by

$$p_{kj}^g = \lambda_{kj}^g (\bar{p}^g - p_k^g),$$

since bank $j$ needs to pay the proportion $d_{kj}$ of the loss $p_k^g$ incurred by bank $k$'s default. We can define $a_i$, the total asset of bank $i$ after the reduction of default cost, as

$$a_i = (1 - c_i \lambda_{ij}(i))e_i + \sum_{k=1}^{n} \sum_{j=1}^{n} m_{ij}^k p_j^k.$$  

We can write in the matrix form

$$a(p; \Delta) = (I - C\Delta) \sum_{k=1}^{n} M^k p^k.$$  

To define the clearing payment vector, we need to describe the seniority structure of debt and CDSs. To this end, we first present the notion of contract payoff function. Let $\phi_j(k)$ be the order function of repayment of bank $j$ as $\phi_j(1) = 1$, bank $j$ repays $p_{k1}$ first; $\phi_j(2) = 2$, bank $j$ repays $p_{k2}$ second; $\ldots$ $\phi_j(n) = n$, bank $j$ repays $p_{kn}$ last. The sum of bank $j$'s repayment that is senior to $d_{kj}(p)$ is written as

$$d_{kj}^g(p) = \lambda^g_{kj} (\bar{p}^g - p_k^g).$$

for $j \neq k$. In other words, $d_{kj}$ is the amount to pay by bank $j$ for the contract of CDS with reference to bank $k$. In addition, we define $\phi_j$ to be the order function of repayment of bank $j$ as $\phi_j(1) = 1$, bank $j$ repays $p_{k1}$ first; $\phi_j(k2) = 2$, bank $j$ repays $p_{k2}$ second; $\ldots$ $\phi_j(kn) = n$, bank $j$ repays $p_{kn}$ last. The sum of bank $j$'s repayment that is senior to $d_{kj}(p)$ is written as

$$d_{kj}^g(p) = \sum_{\phi_j(k') < \phi_j(k)} d_{kj}^g(p).$$

**Definition 1** The vector $p$ and matrix $\Delta$ are a clearing payment vector and a clearing default matrix, respectively, if

$$p_0 = \left( a(p; \Delta) - d_{kj}^0(p) \right)_+$$

$$p^k = \left( a(p; \Delta) - d_{kj}^k(p) \right)_+ \land d_{kj}^g(p)$$

where

$$C = \text{diag}(f_i, i = 1).$$
where \( mk \)
\[ ii = 0 \] for \( i = 1; \ldots; n \). Note that \( Mk \)
includes the cross-ownership structure of debt by
\[ mkik. \]
for \( k = 1; \ldots; n \). The pair \(( p; \Delta)\) is said to be a

**clearing system.**

The clearing payment vector \( p \) is expressed as \( p = f(p; \Delta); \)
\( (3) \) where vector function \( f \) is naturally de ned with \( (1) \) and
\( (2) \). Expression \( (3) \) implies that the clearing payment vector
is a solution of the _xed point problem given \( \Delta. \)

**III. Existence of a Clearing System**

In [3] and other related studies, the clearing default matrix
is automatically determined by the conditions

\[
1_D(\hat{n}) = 1\{a_i < d_{ij}^0\}(\hat{n}),
\]

In other words, bank \( i \) will default if the total liability _\( d_{0i} \)
exceeds the total asset for the repayment _\( ai. \) However, in
our case where CDSs are cross-held among banks, a
clearing payment vector might not exist, since function \( f \) is
neither increasing nor continuous in \( p. \) Therefore, we
propose the notion of a _ctitious default algorithm with
ancial covenants, which describes the definition of
default in our model.

**Assumption 2** At maturity, the clearing default
matrix is determined as follows.

1. Set \( \Delta(0) = 0. \)
2. For \( \ell \)-th step

   i. calculate \( p(\ell) \) satisfying \( p(\ell) = f \)
   \[
   (p(\ell); \Delta(\ell-1))
   \]
   ;

   ii. set \( D(\ell) = D(\ell-1) [f_i 2 f]; : : : ; ngjp0i \)
   \[
   = 0g; \text{ and}
   \]
   iii. update \( \Delta(\ell) = \text{diag}(fD(\ell)(i)gn1) \)
   \[
   = 1).
   \]

2. Stop when \( \Delta(\ell) = \Delta(\ell-1) \) and set \( (p; \Delta) = (p(\ell); \Delta(\ell)). \)

Now we are ready to present the main theorem.

**Theorem 1** There exist a clearing system exists
under Assumptions 1 and 2.

**Proof** Given a _xed default matrix \( \Delta, \) the vector function \( f \)
is continuous and contraction mapping in \( \ell 1 \)-norm owing to
Assumption 1, implying that \( f \) has a _xed point \( p = f(p; \Delta). \)
In the algorithm stated in Assumption 2, the \( n - n \) matrix
sequence \( \Delta(\ell) \) is increasing and has an upper bound \( I, \)
meaning that \( \Delta(\ell) \) converges at most \( n \)steps. Now the
theorem readily follows. □

**IV. Numerical Analysis**

In this section, we implement numerical calculations to
demonstrate how the cross-transaction of CDSs have an impact on
market stability. Following Merton-type structural models,
we suppose that the business assets of bank \( i \) at maturity are
given by

\[
e_i = e_0 \exp\left\{ \left( \mu - \frac{\sigma^2}{2} \right) \ell + \sigma \omega(\ell) \right\},
\]

where _\( \mu \) is the expected growth rate, _\( \sigma \) the volatility, and \( f_i \)
a random term following a standard normal. The \( n \)-dimensional random vector \( e = (e_1; : : : ; en)^T \) is assumed
to be symmetrically distributed with correlation coefficient
It is worth noting that the interconnectedness of CDS transactions has two dimensions. The first dimension is the number of names of CDSs that each bank writes, and the second is the number of banks that each CDS name is owned by. To study the issue effectively, we consider the market in which the network structure is symmetric among banks. We say that the market is of type- $(\ell_1; \ell_2; \ell_3)$ for $\ell_1; \ell_2; \ell_3 \geq 9$, if each debt is owned by $\ell_1$ banks, each bank writes CDSs of $\ell_2$ names (other banks), and each CDS is owned by $\ell_3$ banks. Specifically, we set:

$$
\lambda_j^k = 1_{\{j = k\}} + \frac{1}{n} \times 1 \left\{ j \in \bigcup_{h=1}^{\ell_1} \{ \text{mdl}^k_n + h \} \right\},
$$

$$
m_{ij}^k = \frac{1}{n} \times 1 \left\{ j = k, i \in \bigcup_{h=1}^{\ell_2} \{ \text{mdl}^k_n + h \} \right\},
$$

$$
\phi_j^k (k) = \text{mdl}^k_n + 1 - j,
$$

where

$$\text{mdl}_k^n = f_1 f_2 f_1 \cdots n g j_i = k (\text{mod} \ n),$$

The parameter values in the base case setting are presented in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>number of banks</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{p}$</td>
<td>face value of debt</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>capital conservation buffer</td>
<td>0.2</td>
</tr>
<tr>
<td>$c_i$</td>
<td>default cost ratio</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu$</td>
<td>growth rate of assets</td>
<td>0.05</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>volatility of assets</td>
<td>0.5</td>
</tr>
<tr>
<td>$\rho$</td>
<td>correlation of assets</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 1: Base case parameter setting.

First, we present the result in the case in which only debt is cross-held among banks. Figure 1 depicts the probabilities of how many banks default in the market of type-$(\ell_1; 0; 0)$.

Fig. 1: The effect of connections via debt-holding in the market with no CDS. It is observed from Figure 1 that if the connection is complete in the sense of graph theory, the probability of a default of a bank occurring becomes zero and the number of defaulting banks is always zero in our simulations. This implies that the significantly strong cross-holdings of debt reduce the occurrence of default contagion. This result has already been found by many studies, e.g., [1] and [2]. In summary, the cross-holding of debt has a positive effect on market stability when banks cross-hold only their debt. Now, we consider the effect of CDS cross-writing. Figure 2 describes the probabilities in the market of type-$(6; \ell_2; 6)$.

Fig. 2: The effect of connections via CDS-writing in the market of type-$(6; \ell_2; 6)$.
In contrast to Figure 1, both the probability that no banks default and the probability that all banks default become higher as $\ell^2$ takes a higher value. This means that a strong connection among banks via CDS-writing might lead to an extreme result, namely, all survive or all default. Finally, Figure 3 shows how the probabilities behave, depending on the connectedness of CDS-holding.

V. Conclusion

This study analyzes the effect of the cross-trading of CDSs among banks on market stability and systemic risk. We assume that the default of a bank is determined by a fictitious default algorithm with financial covenants, which guarantees the existence of a clearing system. Our numerical results show that the strong cross-writing or cross-holding of CDSs might lead to market instability and a serious systemic risk. This observation is in stark contrast to the case of cross-holding of debt, in which the complete network always improves market stability. It should be emphasized that the current study is the first to present a theoretical result in which systemic risk might arise when the complete network among banks is present with an $[3]$-type model.

VI. References